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STATISTICAL MODELING OF THE TURBULENT TRANSITION IN THE BOUNDARY LAYER

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A method of statistical modeling the flow in the boundary-layer transition region is proposed on the basis of experimental data on kinematics and dynamics of turbulent spots (Emmons spots) on a flat plate in an incompressible fluid. This method allows one to determine the intermittency with allowance for overlapping of the spots, the forces on the plate surface, and the flow field in the vicinity of the transition region if the field of the streamwise component of the mean velocity in the developed turbulent boundary layer is known as a function of the Reynolds number. In contrast to multi-parameter models of the transition, this approach makes it possible to avoid the use of physically meaningless parameter values.

Key words: *boundary layer, laminar–turbulent transition, statistical modeling.*

Introduction. There are some recent attempts to study turbulent flows by statistical modeling methods [1, 2]. Models that can be readily interpreted within the framework of the probability theory are used [3]. In the problem of the laminar–turbulent transition, such a model is the model of the emergence and evolution of turbulent spots proposed by Emmons [4]. This approach is based on experimental observations of the emergence of spots (seeds of turbulence) at a certain critical value of the Reynolds number; these spots follow known simple laws when they grow in the downstream direction, preserving their shape. The flow characteristics inside the spots are close to the characteristics of a developed turbulent flow with the Reynolds number corresponding to the spot location, which allows the mean forces and fields in the transition region to be determined. In the vicinity of the critical Reynolds number, the spots appear randomly in space and time. Using information about the geometry of the spots and statistical data on their emergence, Emmons [4] determined the probability of a certain point on the plate surface to be covered by a turbulent spot, i.e., the intermittency. The spots were assumed to be not overlapping one another. Using the method of statistical modeling, one can easily obtain a similar picture even with allowance for spot overlapping if the kinematics and dynamics of individual spots and the character of their interaction are known. Such data were obtained in some recent experimental studies (see, e.g., [5–7]) whose results are used in the present work.

To close the model, one should know the critical value of the Reynolds number at the beginning of the transition. There are numerous experimental data (see, e.g., [8]) on the relation between the free-stream parameters and the transition Reynolds number $\mathrm{Re}_{\mathrm{cr}}$. It seems reasonable to use this information at the initial stage of simulations. Nevertheless, some attempts were made not only to solve this problem phenomenologically (with the use of the criterion e^n), but also to determine $\mathrm{Re}_{\mathrm{cr}}$ with allowance for the evolution of disturbances in the region upstream of the transition. Rubinstein and Choudhari [9] predicted this value on the basis of the wave evolution statistics in the three-wave resonance approximation. Dodonov et al. [10] and Zharov [11] proposed methods of determining the wave-packet dynamics, which can also be used for calculating $\mathrm{Re}_{\mathrm{cr}}$. It should be noted that there are numerous attempts to determine the transition region with the help of multi-parameter models of turbulence. The practice of such calculations shows, however, that non-physical values of the governing parameters of such models have to be specified to calculate a realistic value of $\mathrm{Re}_{\mathrm{cr}}$ [12].

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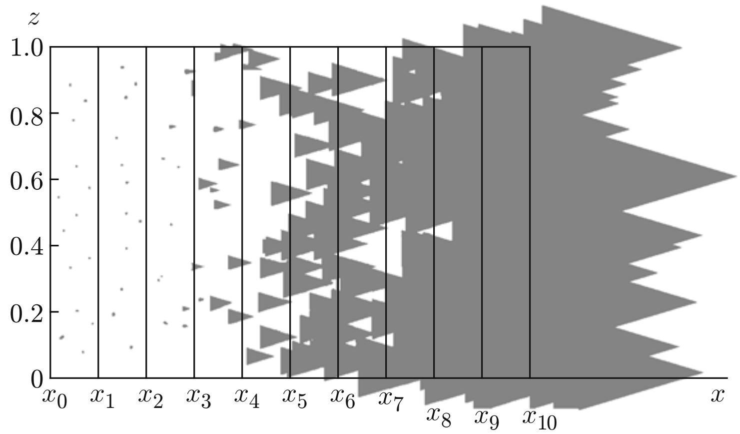


Fig. 1. Kinematics of turbulent spots at $\lambda = 200$ ($x_{10} = 1$).

1. Formulation of a Probabilistic Model of the Transition. The following method is used to determine the probability density function of flow turbulization in the transition region. Beginning from the value $\mathrm{Re}_{\mathrm{cr}}$ (which is reached at the streamwise coordinate x_0), point spots are formed in the flow region with a frequency λ in accordance with the probability density distribution in time $\rho(t) = \lambda \exp(-\lambda t)$. The spots are formed statistically uniformly in the region $z \in [0, 1]$, $x \in [0, x^*]$. The value of x^* in the calculations was assumed to be 0.1–0.2. Moving downstream, the spots start growing (Fig. 1). The code modeling the spot evolution in time and space was written with the use of the "Mathematica" software system. Figure 2 shows the evolution of the characteristic points (P_0, P_1, and P_2) of the turbulent spot, which are used in this code. The spot shape considered in the present work is a isosceles triangle (this shape remains unchanged during the entire observation time); the base of this triangle moves with a velocity $V_{P_0} = 0.5U_\infty$, the vertex moves with a velocity $V_{P_1} = 0.89U_\infty$, and the side vertex moves with a velocity $V_{P_2} = 0.1U_\infty$. We consider the region between the line $x_0 = 0$ (see Fig. 1), where the spots emerge, and the line $x_l = l$, where the spots density is so high that the flow as a whole can be considered as turbulent. The line $x_l = l$ is determined from a numerical experiment. The flow domain $x \in [0, l]$ is divided into n identical subdomains (in our calculations, we used $n = 10$). Each subdomain is randomly filled by stochastically uniformly distributed points (in our calculations, 5000 points in each subdomain). After that, the number $N_s/N = f(x)$ is determined (N_s is the number of points inside the spots and N is the total number of spots in the subdomain). Occurrence of a point in the spot, i.e., occurrence of a point (x, z) in the domain occupied by the spot is determined by the condition

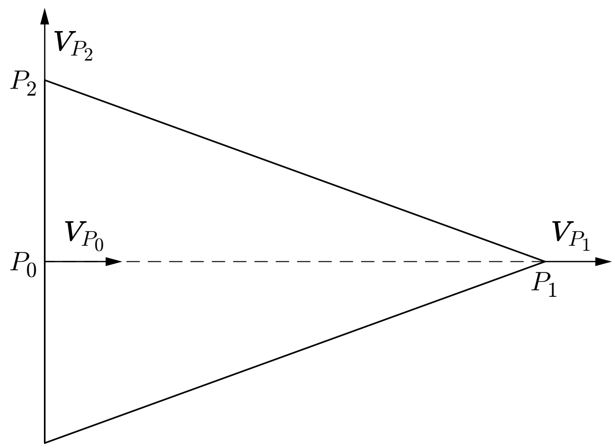
$$q(t, x, z, \boldsymbol{s}) = \frac{|-V_{P_0}t + x - s_x|}{(V_{P_1} - V_{P_0})t} + \frac{|z - s_z|}{V_{P_2}t} \leqslant 1 \ \cap \ x - V_{P_0}t - s_x \geqslant 0,$$

where $\boldsymbol{s} = (s_x, s_z)$ are the coordinates of the point P of the spot. The condition for a point to belong to at least one spot can be written as

$$Q = \bigcup_{i=1}^{N} q(t, x, z, \boldsymbol{s}^i).$$

The values of f obtained are then averaged over several realizations.

The code for determining the probability density function of flow turbulization was developed on the basis of the "Mathematica" software system. Figure 3 shows the spots inside a chosen band. Note that the probability density function of flow turbulization (with a dimensionless area S_s/S) is determined in the present work with allowance for overlapping of the spots, in contrast to [4]. The probability density function f of flow turbulization in the transition region at $\lambda = 200$ (points) is plotted in Fig. 4. This figure also shows the curve $f_1 = 0.5(1 + \operatorname{erf}((x - a)/b))$ obtained by the least squares method, which is a good approximation of the numerical results. For comparison, the same method was used to determine the constant in the expression $f = 1 - \exp(-x^2/x_0^2)$ derived theoretically in [4]. It is seen in Fig. 4 that the function of such a form ensures a worse approximation of the numerical data. The quantities a and b are determined as functions of λ through comparisons of the numerical data



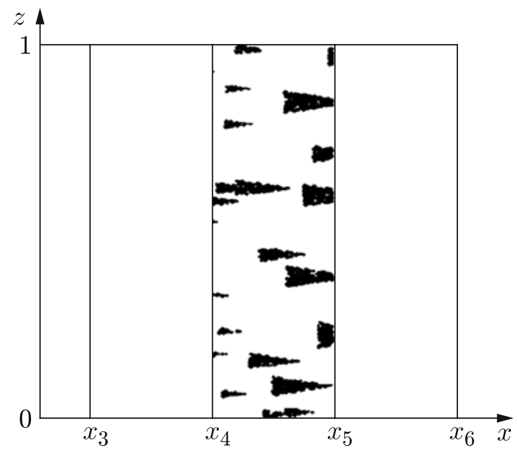
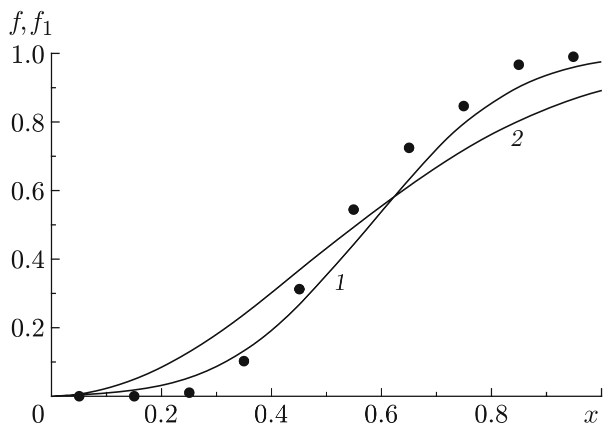


Fig. 2 Fig. 3

Fig. 2. Geometry of the turbulent spot and velocities of motion of its characteristic points P_0, P_1, and P_2.

Fig. 3. Determination of the area occupied by the spots in a chosen subdomain by the method of statistical modeling.



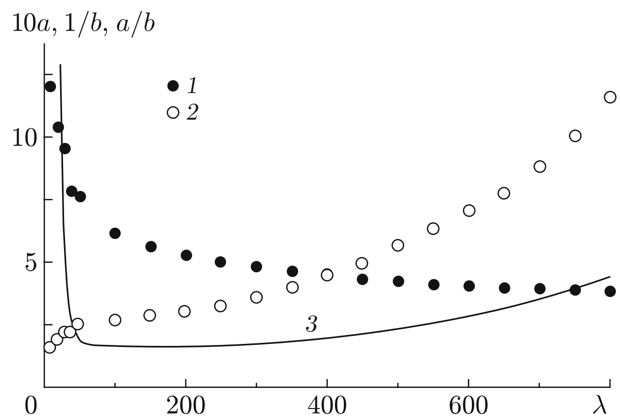


Fig. 4 Fig. 5

Fig. 4. Probability density function f of flow turbulization in the transition region: 1) $f_1 = 0.5(1 + \mathrm{erf}\,((x-a)/b)$; 2) $f = 1 - \exp(-x^2/x_0^2)$ [4]; the points are the results calculated at $\lambda = 200$ in the present work.

Fig. 5. Coefficients a, $1/b$, and a/b as functions of the rate of emergence of turbulent spots λ (points 1 and 2 refer to a and $1/b$, respectively, and curve 3 refers to a/b.

with the results of approximation (Fig. 5). The quantity x is dimensionless, i.e., it is normalized to a certain reference length L. The length L is also used for normalization over the transverse coordinate. If the length is measured in Reynolds numbers, we obtain the Reynolds numbers Re and Re_0 based on these lengths: $x/L = \mathrm{Re}\,/\,\mathrm{Re}_0$.

2. Determination of the Drag Coefficient of the Flat Plate with the Transition. The local drag coefficient of the flat plate is determined by the formula

$$c'_{\mathrm{f}} = (1 - f_1(\mathrm{Re}))c'_{\mathrm{f,lam}} + f_1(\mathrm{Re})c'_{\mathrm{f,turb}}.$$

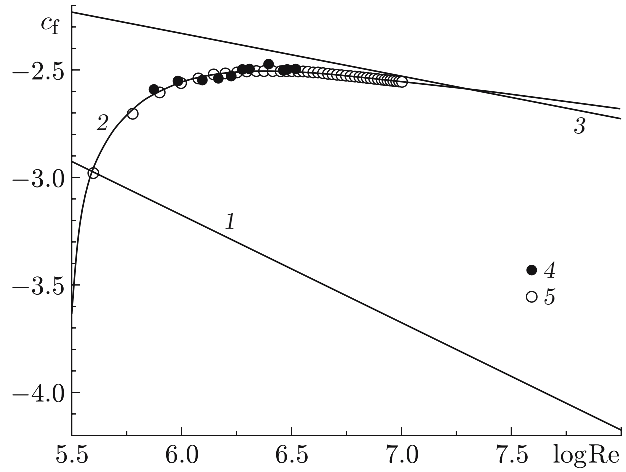


Fig. 6. Total drag coefficient of the flat plate versus the Reynolds number of the plate: curve 1 refers to $c_{\mathrm{f,lam}} = 0.664/\,\mathrm{Re}_l^{3/2}$ [8] (the drag of the plate in a laminar flow); curve 2 refers to $c_\mathrm{f} = 0.455/\log{(\mathrm{Re}_l)}^{2.58} - 1700/\,\mathrm{Re}_l$ [8] (the drag of the plate with allowance for the laminar part of the flow near the leading edge), and curve 3 refers to $c_\mathrm{f} = 0.074/\,\mathrm{Re}_l^{1/5}$ [8] (the drag of the plate in a turbulent flow); points 4 and 5 are the Herbers' experimental results [8] and the results calculated in the present work, respectively.

(The curve of this dependence should coincide with the curve obtained in the experiment [8].) Then, the expression for the integral drag coefficient can be written as

$$c_\mathrm{f} = \frac{1}{\mathrm{Re}_l}\int\limits_0^{\mathrm{Re}_l} c'_\mathrm{f}\,d\,\mathrm{Re}_l = \begin{cases} \dfrac{1}{\mathrm{Re}_l}\displaystyle\int\limits_0^{\mathrm{Re}_l} c'_{\mathrm{f,lam}}\,d\,\mathrm{Re}_l, & \mathrm{Re}_l < \mathrm{Re}_k; \\ \dfrac{1}{\mathrm{Re}_l}\displaystyle\int\limits_0^{\mathrm{Re}_k} c'_{\mathrm{f,lam}}\,d\,\mathrm{Re}_l + \dfrac{1}{\mathrm{Re}_l}\displaystyle\int\limits_0^{\mathrm{Re}_k} ((1-f_1)c'_{\mathrm{f,lam}} + f_1 c'_{\mathrm{f,turb}})\,d\,\mathrm{Re}_l, & \mathrm{Re}_l \geqslant \mathrm{Re}_k \end{cases}$$

or

$$c_\mathrm{f} = \begin{cases} \dfrac{1.328}{\sqrt{\mathrm{Re}_l}}, & \mathrm{Re}_l < \mathrm{Re}_k; \\ \dfrac{\mathrm{Re}_k}{\mathrm{Re}_l}\,\dfrac{1.328}{\sqrt{\mathrm{Re}_l}} + \dfrac{1}{\mathrm{Re}_l}\displaystyle\int\limits_{\mathrm{Re}_k}^{\mathrm{Re}_l}\Big((1-f_1)\,\dfrac{0.664}{\sqrt{\mathrm{Re}}} + f_1\,\dfrac{0.0576}{\sqrt{\mathrm{Re}}}\Big)\,d\,\mathrm{Re}, & \mathrm{Re}_l \geqslant \mathrm{Re}_k\,. \end{cases}$$

The expression for f_1 can be written in a more convenient form as

$$f_1 = 0.5[1 + \mathrm{erf}\,(a(\mathrm{Re} - \mathrm{Re}_k - b\,\mathrm{Re}_0)/\,\mathrm{Re}_0)].$$

Here Re is the local Reynolds number of the point of the flat plate and Re_0 and Re_k are auxiliary coefficients. As we consider the boundary layer on the plate with a zero streamwise pressure gradient, we can determine the values of a, b, Re_0, and Re_k by comparing the function c_f with the experimental data [8]. First, we determine the dependence of the ratio a/b on λ (see Fig. 5). Note that this dependence has a minimum at the point $\lambda = 132$. We use this value of λ in our calculations, because the transition model considered seems to be inapplicable at lower values of λ, and the maximum value of the transition Reynolds number obtained in the experiment corresponds to the value $\lambda = 132$. (Note that the effective width of the transition region remains practically constant with decreasing λ, it is only the Reynolds number of the transition beginning that increases, which disagrees with the scenario of the transition region development.) The values of λ and Re_k will be different for different free-stream parameters. In this case, the transition description reduces to determining the dependence of the parameters λ and Re_k on the free-stream conditions (for instance, on the amplitude and spectral composition of disturbances on the leading edge).

Determining a and $1/b$ at the point $\lambda = 132$ in accordance with approximations with respect to λ, we obtain $a = 0.55$ and $1/b = 2.73$. The value of Re_k is determined from the value of the Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ corresponding to the beginning of the transition. The value of Re_0 is chosen from the condition of the best agreement between the values of c_{f} and experimental data [8]. As a result, we obtain $\mathrm{Re}_0 = 10^4$ and $\mathrm{Re}_k = 4 \cdot 10^5$. Figure 6 shows the drag coefficient of the plate on the Reynolds number, which agrees well with the experimental results reported in [8].

Conclusions. Thus, the use of statistical modeling methods makes it possible to determine the kinematics of turbulent spots in the transition region on a flat plate at different times. Knowing the frequency of emergence of the spots λ, one can determine the local degree of flow turbulization, which allows one to calculate the turbulent flow characteristics in the transition region, for instance, local or integral forces, or to construct the mean flow field in the transition region. In contrast to [4], the degree of flow turbulization here is determined with allowance for overlapping of the spots. The fact that the dependence of a/b on λ has a minimum, apparently, means that the model of the transition induced by the development of turbulent spots is inapplicable at lower values of λ.

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